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The Generating Functional for the Probability Density Functions of Navier–Stokes Turbulence

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A generating functional for the equal-time spatial probability density functions which represent the ensemble of turbulent incompressible Navier–Stokes fluids is introduced. By formally solving the linear evolution equation satisfied by this functional, the probability densities are represented as functional integrals. It is shown that the generating functional can be regarded as the space characteristic functional of a generalized random field defined on the phase space spanned by the material position and velocity fields of a fluid particle. The interpretation of this random field, which satisfies a dynamical equation similar to Vlasov's, is clarified through the formal analogies between the statistics of molecules and fluid particles at the functional level. A class of statistically realizable and solvable models is also considered within the context of the present formalism.

KEY WORDS: Turbulence; probability density function; generating functional; Hopf's equation; microscopic phase density; Vlasov's equation; functional integration; Bogolyubov's functional formalism.

1. INTRODUCTION

As in the case of interacting molecules, the finite-dimensional probability density function (p.d.f.) approach to turbulent flows leads to closure problems. Due to viscous foces and the nonlocal relation between pressure and velocity fields, the evolution equation for the n-point p.d.f. f_n associated with the spatial (Eulerian) velocity field $u(x, t)$ cannot be expressed in terms of f_n alone for any $n \geqslant 1$; it involves both f_n and f_{n+1}. The single-time n-point spatial p.d.f. f_n is a function of $6n+1$ variables and is defined by

$$f_n(v_1, x_1, \ldots, v_n, x_n, t)\, dv_1 \cdots dv_n = \text{Prob}\left\{ \bigcap_{k=1}^{n} v_k < u(x_k, t) < v_k + dv_k \right\} \qquad (1)$$

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where x_k and v_k are the position and velocity vectors associated with point k in the fluid at time t, respectively. The coupled sequence of dynamical equations for these p.d.f.'s, which manifest the closure problem, appeared for the first time in the work of Lundgren[1] and Monin,[2] and the corresponding equivalent hierarchy for the vorticity field $\nabla \times u$ was presented by Novikov.[3] The motions of isolated boundary-free incompressible Newtonian fluids with uniform density ρ and kinematic viscosity ν are well described by the Navier–Stokes equation

$$\frac{\partial u^\alpha}{\partial t} + u^\beta \frac{\partial u^\alpha}{\partial x^\beta} = \nu \,\Delta u^\alpha - \frac{1}{4\pi} \int_{R^3} dx_1 \left(\frac{\partial}{\partial x^\alpha} \frac{1}{|x - x_1|} \right) \frac{\partial^2 u_1^\beta u_1^\gamma}{\partial x_1^\beta \, \partial x_1^\gamma} \tag{2}$$

with a prescribed initial state $u(x, t_0) \equiv u_0(x)$ which satisfies $\nabla \cdot u_0 = 0$; here, $u_1 \equiv u_1(x_1, t)$, superscripts specify the vector components, and a sum over the repeated Greek indices is implied. By adopting the assumption that each realization in the ensemble of turbulent flows evolves according to (2), one derives a coupled hierarchy for p.d.f.'s (the LMN hierarchy) as

$$\frac{\partial f_n}{\partial t} + \sum_{k=1}^{n} \left\{ v_k^\alpha \frac{\partial f_n}{\partial x_k^\alpha} + \frac{\partial}{\partial v_k^\alpha} \int dx_{n+1} \, dv_{n+1} \, F_\alpha(x_k - x_{n+1}, v_{n+1}) f_{n+1} \right\} = 0 \tag{3}$$

where $n = 1, 2, \ldots$ and the function

$$F_\alpha(x - y, v) = -\frac{1}{4\pi} \left(v^\beta v^\gamma \frac{\partial^3}{\partial x^\alpha \, \partial x^\beta \, \partial x^\gamma} \frac{1}{|x - y|} + \nu v^\alpha \Delta^2 \frac{1}{|x - y|} \right) \tag{4}$$

expresses the pressure and viscous forces between fluid particles. Here, for convenience, the statistics of turbulence ensemble at time $t > t_0$ is presumably determined by the prescribed statistics of $u_0(x)$ only. In addition to (3), the p.d.f.'s also satisfy various consistency and incompressibility conditions.[4] Although the temporal and spatial scales associated with fluid particles which represent a turbulent medium are not similar to those of actual molecules, and the interaction force (4) is dissipative and velocity dependent, it is possible to exploit the formal similarity between (3) and the BBGKY hierarchy to some extent.[5-8]

The closure problem associated with (3) can formally be avoided by incorporating the entire family of p.d.f.'s into a functional. Clearly, a variable which carries all the information contained in the infinite sequence $\{f_n \mid n \geqslant 1\}$ for all times is statistically equivalent to the probability measure associated with $u(x, t)$. This paper presents a functional formulation which is related to the p.d.f. hierarchy of Navier–Stokes turbulence theory in the similar way that Bogolyubov's functional formalism[9] is related to the molecular BBGKY hierarchy. It turns out that this formalism is structurally connected to the LMN hierarchy in the *same* way that Hopf's original formulation[10] is related to the hierarchy of moments of $u(x, t)$. In Section 2, by using material (Lagrangian) flow variables, a functional which generates the p.d.f.'s of $u(x, t)$ is introduced and a closed, second-order linear evolution equation for this functional is presented and the equation is reinterpreted as the Hopf equation associated with a generalized scalar random field defined on the six-dimensional phase space which is spanned by the material position and velocity fields of a fluid particle. This field statistically plays the role of a microscopic phase density for fluid particles and specifies the Navier–Stokes ensemble completely. Section 3 contains the integral representations of p.d.f. generating functional and n-point p.d.f. Some exactly solvable and realizable fluid dynamical models and analogies to the molecular systems are also presented in this section. In concluding in Section 4, various generalizations are briefly mentioned.

2. PDF GENERATING FUNCTIONAL AND ITS INTERPRETATION

Let $r(a, t)$ and $v(a, t) \equiv (\partial/\partial t)\, r(a, t)$ be the material position and velocity fields, respectively, of a fluid particle which is identified with its initial position $r(a, t_0) = a$ at time t_0. We have

$$v(a, t) = u(r(a, t), t) \tag{5}$$

and for incompressible fluids $\det(\partial r^\alpha/\partial a^\beta) = 1$ or, equivalently, $\nabla \cdot u = 0$ for all $t > t_0$. Now consider the following complex-valued functional:

$$G[\eta(x, v), t] = \left\langle \exp\left\{ i \int_{R^3} da\, \eta(r(a, t), v(a, t)) \right\} \right\rangle \tag{6}$$

where $\eta(x, v)$ is a real time-independent test function defined on Euclidean space R^6, that is, on the phase space of a single fluid particle. Here, the bracket $\langle \cdots \rangle$ indicates an averaging over the ensemble of Navier–Stokes flows at time t or, in general, an integration over the measure associated with random fields inside the bracket. $G[\eta, t]$ incorporates all the statistical information contained in the infinite sequence of f_n. Indeed, from (5) and (6) one can write

$$G[\eta, t] = \left\langle \exp\left\{ i \int_{R^3} dx\, \eta(x, u(x, t)) \right\} \right\rangle \tag{7}$$

by applying the incompressibility condition, and consequently

$$G[\eta, t] = \sum_{n \geq 0} \frac{i^n}{n!} \int \cdots \int dw_1 \cdots dw_n\, f_n(w_1, \ldots, w_n, t)\, \eta(w_1) \cdots \eta(w_n) \tag{8}$$

where and in the following we set $w \equiv (x, v)$.

$G[\eta, t]$ defined as (6) contains all the statistical information about $u(x, t)$ at a given time; however, (6) does not give the multiple-time statistics. From a prescribed initial statistical state $G[\eta, t_0] = G_0[\eta]$, $G[\eta, t]$ is evidently determined by the dynamics of $u(x, t)$. Therefore, it directly follows from (2), (4), and (7) that

$$\frac{\partial G}{\partial t} = -\int dw_1\, \eta(w_1) v_1^\alpha \frac{\partial}{\partial x_1^\alpha} \frac{\delta G}{\delta \eta(w_1)} + i \iint dw_1\, dw_2\, \eta(w_1)\, F_\alpha(w_1, w_2) \frac{\partial}{\partial v_1^\alpha} \frac{\delta^2 G}{\delta \eta(w_1)\, \delta \eta(w_2)} \tag{9}$$

where $F_\alpha(w_1, w_2) \equiv F_\alpha(x_1 - x_2, v_2)$ is the same pressure–viscosity kernel defined by (4). Equation (9), which governs the statistical dynamics of boundary-free incompressible fluids, also follows at once from the general Liouville-type equation[11] governing the statistics of arbitrary functionals of $u(x, t)$.

Formally, one may rewrite Eq. (9) as

$$\frac{\partial G}{\partial t} = i \int dw_1\, \eta(w_1)\, V\left[\frac{\delta}{i\delta\eta}\right] G \tag{10}$$

where

$$V\left[\frac{\delta}{i\delta\eta}\right] = -v_1^\alpha \frac{\partial}{\partial x_1^\alpha} \frac{\delta}{i\delta\eta(w_1)} - \int dw_2\, F_\alpha(w_1, w_2) \frac{\partial}{\partial v_1^\alpha} \frac{\delta}{i\delta\eta(w_1)} \frac{\delta}{i\delta\eta(w_2)} \tag{11}$$

and attempt to interpret (10) as the Hopf equation[10] associated with a scalar random field $g(w, t)$ whose dynamics is governed by $\partial g/\partial t = V[g]$ or, explicitly,

$$\frac{\partial g}{\partial t} + v^\alpha \frac{\partial g}{\partial x^\alpha} + \frac{\partial g}{\partial v^\alpha} \int dw_1\, F_\alpha(w, w_1)\, g(w_1, t) = 0 \tag{12}$$

Evidently not all solutions of Eq. (10) are characteristic functionals of probability measures. However, from the definition (6) of $G[\eta, t]$ we have (i) $G[0, t] = 1$, (ii) G is continuous in η, and (iii) $G[\eta, t]$ is nonnegative definite, *i.e.*, $\sum_{k,l} c_k \bar{c}_l G[\eta_k - \eta_l, t] \geqslant 0$ for the arbitrary sets of n functions $\{\eta_k \mid 1 \leqslant k \leqslant n\}$ and n (complex) numbers $\{c_k \mid 1 \leqslant k \leqslant n\}$ with $n = 1, 2,\ldots$. Thus, $G[\eta, t]$ is the characteristic functional of a countably additive, positive, and normalized probability measure defined on the dual of a nuclear test function space.[12] Consequently, there exists a generalized random field $g(\eta) \equiv (g, \eta) \equiv \int dw\, g(w, t)\, \eta(w)$ such that

$$G[\eta, t] = \int e^{ig(\eta)}\, d\gamma_t(g) \equiv \Phi_g[\eta, t] \tag{13}$$

where $d\gamma_t(g)$ represents the measure associated with $g(w, t)$. Therefore, on the space of functions $\eta(w)$, the p.d.f. generating functional of $u(x, t)$ formally coincides with the characteristic functional of $g(w, t)$; accordingly, we have

$$\left\langle \prod_{k=1}^{n} g(w_k, t) \right\rangle = f_n(w_1,..., w_n, t) \qquad \text{for all} \quad n \geqslant 1$$

Observe that $g(w, t)$ and $u(x, t)$ specify the physical ensemble of turbulent flows at the same statistical information level; thus, the entire statistical turbulence dynamics of incompressible viscous fluids can be based on Eq. (12), which is formally similar to Vlasov's equation [see ref. 13 for equations similar to (12)]. Since the moments of $g(w, t)$ must be positive and normalized with respect to v integrations over R^3 for arbitrary sets of noncoincidental phase points, one sets $g(w, t) \geqslant 0$ and $\int_{R^3} dv\, g(w, t) = 1$ for (almost) all realizations of $g(w, t)$.

It follows from (7) and the incompressibility condition $\nabla \cdot u = 0$ that

$$G[\eta + v \cdot \nabla\phi, t] = G[\eta, t] \tag{14}$$

where $\phi = \phi(x)$ is a test function of spatial coordinates only, or equivalently

$$\frac{\partial}{\partial x^\alpha} \int dv\, v^\alpha \frac{\delta G}{\delta \eta(w)} = 0 \tag{15}$$

The condition (15) clearly implies

$$\frac{\partial}{\partial x_k^\alpha} \int dv_k\, v_k^\alpha f_n(w_1,..., w_n, t) = 0 \tag{16}$$

for $1 \leqslant k \leqslant n$ and $n = 1, 2,...$. Since the Navier–Stokes dynamics preserves the solenoidal character of $u_0(x)$, it is sufficient to invoke (14) or (15) for the initial state $G_0[\eta]$ only. Similarly, the consistency of solutions to Eq. (12) with incompressibility at all times implies

$$\frac{\partial}{\partial x^\alpha} \int dv\, v^\alpha g(w, t) = 0 \tag{17}$$

for $t \geqslant t_0$. In addition to (14), $G[\eta, t]$ also satisfies various compatibility relations from which corresponding well-known constraints on p.d.f.'s can be derived.[4] For example, for $1 \leqslant k \leqslant n$, we have

$$\int \cdots \int dv_1 \cdots dv_k \frac{\delta^n G}{\delta\eta(w_1) \cdots \delta\eta(w_n)} = i^k \frac{\delta^{n-k} G}{\delta\eta(w_{k+1}) \cdots \delta\eta(w_n)} \tag{18}$$

which gives the reduction property

$$\int \cdots \int dv_1 \cdots dv_k \, f_n(w_1, \ldots, w_n, t) = f_{n-k}(w_{k+1}, \ldots, w_n, t) \tag{19}$$

with $f_0 = 1$. Moreover, for $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant n$,

$$\lim_{x_k \to x_l} \frac{\delta^n G}{\delta\eta(w_1) \cdots \delta\eta(w_n)} = i\delta(v_k - v_l) \frac{\delta^{n-1} G}{\delta\eta(w_1) \cdots \delta\eta(w_{k-1}) \, \delta\eta(w_{k+1}) \cdots \delta\eta(w_n)} \tag{20}$$

yields

$$\lim_{x_k \to x_l} f_n(w_1, \ldots, w_n, t) = \delta(v_k - v_l) \, f_{n-1}(w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n, t) \tag{21}$$

The LMN hierarchy (3) can now be derived from (9) by just taking functional derivatives, and possible closure approximations associated with p.d.f.'s may be represented as constraints on $G[\eta, t]$. The quantity $\ln G[\eta, t]$ generates the correlation density functions $C_n(w_1, \ldots, w_n, t)$ of $u(x, t)$, which are defined by the familiar relations

$$\begin{aligned}
C_1(w_1, t) &= f_1(w_1, t) \\
C_2(w_1, w_2, t) &= f_2(w_1, w_2, t) - f_1(w_1, t) \, f_1(w_2, t) \\
C_3(w_1, w_2, w_3, t) &= f_3(w_1, w_2, w_3, t) - f_1(w_1, t) \, f_2(w_2, w_3, t) \\
&\quad - f_1(w_2, t) \, f_2(w_1, w_3, t) \\
&\quad - f_1(w_3, t) \, f_2(w_1, w_2, t) + 2 f_1(w_1, t) \, f_1(w_2, t) \, f_1(w_3, t)
\end{aligned} \tag{22}$$

etc., such that

$$\ln G[\eta, t] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \cdots \int dw_1 \cdots dw_n \, C_n(w_1, \ldots, w_n, t) \, \eta(w_1) \cdots \eta(w_n) \tag{23}$$

The dynamical equation for $\ln G[\eta, t]$ is nonlinear and leads to a nonlinear hierarchy for the C_n; however, the statistical information contained in C_n is not as redundant as that carried by f_n, and the C_n's permit more explicit closure approximations due to their cluster properties. Evidently, the C_n's can be regarded as the (specific) cumulants of $g(w, t)$.

Finally, note that Hopf's original characteristic functional[10] for $u(x, t)$ is a particular restriction of $G[\eta, t]$. On the subset $\{\eta(w) \mid \eta(w) \equiv v \cdot \theta(x)\}$, where $\theta(x)$ is a spatial test vector field, one has

$$G[\eta, t]|_{\eta = v \cdot \theta} = \left\langle \exp\left\{ i \int dx\, \theta(x) \cdot u(x, t) \right\} \right\rangle \equiv \Phi_u[\theta, t] \tag{24}$$

and

$$\int \cdots \int dv_1 \cdots dv_n\, v_1^{\alpha_1} \cdots v_n^{\alpha_n} \left. \frac{\delta^n G}{\delta\eta(w_1) \cdots \delta\eta(w_n)} \right|_{\eta = v \cdot \theta} = \frac{\delta^n \Phi_u}{\delta\theta^{\alpha_1}(x_1) \cdots \delta\theta^{\alpha_n}(x_n)} \tag{25}$$

3. FORMAL SOLUTIONS, MODELS, AND INTERPRETATION OF $g(w, t)$

The general solution of the initial value problem associated with Eq. (9) admits an integral representation

$$G[\eta, t] = \int G_0[\eta_0]\, d\mu(\eta_0, t_0; \eta, t) \tag{26}$$

where $\mu(\eta_0, t_0; \eta, t)$ is the (generalized) Green's measure from a statistical state at t_0 to the state at $t > t_0$. The limit form of the iterated composition property of Green's measures for small time intervals[4,14] gives

$$G[\eta, t) = \iint_{\phi(w, t) = \eta} d[\zeta] d[\phi]\, G_0[\phi(w, t_0)] \times \exp\left\{ i \int_{t_0}^{t} d\tau \int dw\, [\dot{\phi}\zeta + \phi V[\zeta]] \right\} \tag{27}$$

where the functional integrations are over the continuous one-particle time-dependent phase functions $\phi \equiv \phi(w, \tau)$ and $\zeta \equiv \zeta(w, \tau)$, $t_0 \leqslant \tau \leqslant t$, such that $\phi(w, t) = \eta(w)$, $\dot{\phi} \equiv (\partial/\partial\tau)\, \phi(w, \tau)$, and the measure $d[\phi] \sim \prod_{w,\tau} d\phi(w, \tau)$, $t_0 < \tau < t$, formally corresponds to the lattice approximation of phase functions.[4,14] Now, from (27) one derives

$$f_n(w_1, \ldots, w_n, t) = \iint_{\phi(w, t) = 0} d[\zeta]\, d[\phi]\, G_0[\phi(w, t_0)] \left\{ \prod_{k=1}^{n} \zeta(w_k, t) \right\} \times \exp\left\{ i \int_{t_0}^{t} d\tau \int dw\, [\dot{\phi}\zeta + \phi V[\zeta]] \right\} \tag{28}$$

which reveals that the exact determination of a p.d..f. of any order at time t requires the prescription of *all* p.d.f.'s at $t_0 < t$. Note from (11) that $V[\zeta]$ is quadratic in $\zeta(w, \tau)$. Therefore, the integrals over this function can formally be performed by applying the standard procedure[15]; however, one is still left with the problems of evaluating the inverse kernel and functional determinant, which both depend on $\phi(w, \tau)$.

Within the p.d.f. generating functional formulation, a class of solvable simple models for fluid turbulence is provided by

$$G[\eta, t] = G_0 \left[\int dw_1\, K(w_1, t; w, t_0)\, \eta(w_1) \right] \tag{29}$$

where the kernel function $K(w, t; w_1, t_1)$ represents a particular model and satisfies $K(w, t; w_1, t) = \delta(w - w_1)$. Statistical consistency is guaranteed if, for each model, the corresponding random field $g(w, t)$ is governed by a *linear* evoluation equation which admits $K(w, t; w_1, t_1)$ as its Green's function in R^6. Since the nonlinearity characteristics of an equation satisfied by $g(w, t)$ differ from those of the corresponding equation for $u(x, t)$, (29) is not restricted to a dynamics linear in $u(x, t)$. For example, the models defined through

$$\frac{\partial g}{\partial t} + v^\alpha \frac{\partial g}{\partial x^\alpha} + \frac{\partial}{\partial v^\alpha}(H_\alpha g) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x^\alpha} \int dv\, v^\alpha g = 0$$

where $H_\alpha \equiv H_\alpha(x, v, t)$ $(\alpha = 1, 2, 3)$ is sufficiently smooth and otherwise arbitrary, correspond to

$$\frac{\partial u^\alpha}{\partial t} + u^\beta \frac{\partial u^\alpha}{\partial x^\beta} = H_\alpha(x, u, t) \qquad \text{and} \qquad \nabla \cdot u = 0 \qquad (\alpha = 1, 2, 3)$$

Evidently, the p.d.f.'s of turbulent flows specified by (29) are completely determined by only their own initial values as

$$f_n(w_1, ..., w_n, t) = \int \cdots \int dw'_1 \cdots dw'_n K(w_1, t; w'_1, t_0) \times \cdots K(w_n, t; w'_n, t_0)\, f_n(w'_1, ..., w'_n, t_0) \tag{30}$$

for all $n \geqslant 1$.

To find a physical interpretation for $g(w, t)$, consider the formal connection between Bogolyubov's functional[9]

$$\mathscr{L}[\eta, t] = \left\langle \prod_{k=1}^{N} \left[1 + \frac{V}{N} \eta(q_k(t), p_k(t)) \right] \right\rangle_m$$

and the characteristic functional

$$\Phi_M[\eta, t] = \left\langle \exp\left\{ i \int dq\, dp\, \eta(q, p)\, M(q, p, t) \right\} \right\rangle_m$$

of the microscopic one-particle[16] density

$$M(q, p, t) = \frac{V}{N} \sum_{k=1}^{N} \delta(q - q_k(t))\, \delta(p - p_k(t))$$

associated with N identical molecules of mass m confined to a region of volume V. Here η is a real test function, (q_k, p_k) are the state variables of the kth molecule, and $\langle \cdots \rangle_m$ denotes an averaging over the molecular ensemble. We have

$$\mathscr{L}\left[\frac{1}{\bar{v}}(e^{i\bar{v}\eta} - 1), t\right] = \Phi_M[\eta, t] \tag{31}$$

where $\bar{v} = V/N$. As the average number density $\bar{n} \equiv 1/\bar{v} \to \infty$ and $m \to 0$ such that $m\bar{n}$ remains finite, the molecular system behaves more like a "continuum" and (31) becomes $\mathscr{L}[i\eta, t] = \Phi_M[\eta, t]$. In this "fluid limit," $M(q, p, t)$ satisfies Vlasov's equation, which can be regarded as the Klimontovich equation without the discrete particle structure.[17] By comparing this limit case with (13), one can gues that $g(w, t)$ plays the role of a microscopic phase density for fluid particles. To verify this, let us introduce the fluid dynamical versions of $\mathscr{L}[\eta, t]$ and $M(q, p, t)$ for $t \geqslant t_0$ as

$$\mathscr{L}_F[\eta, t] = \lim_{N \to \infty} \left\langle \prod_{k=1}^{N} [1 + \Omega(a_k)\, \eta(r(a_k, t), v(a_k, t))] \right\rangle \tag{32}$$

and

$$D(x, v, t) \equiv D(w, t) = \int_{R^3} da\, \delta(x - r(a, t))\, \delta(v - v(a, t)) \tag{33}$$

respectively. In (32), $\Omega(a_k)$ is the small volume element in R^3 which contains the fluid particle labeled by a_k at time t_0; clearly, to cover each fluid particle, $\Omega(a_k) \to 0$ as $N \to \infty$. From (32) and (33) we have $\mathscr{L}_F[i\eta, t] = \langle i \int dw\, \eta(w)\, D(w, t) \rangle \equiv \Phi_D[\eta, t]$, and from definitions (6) and (32) and relation (13) we also have $\mathscr{L}_F[i\eta, t] = G[\eta, t] = \Phi_g[\eta, t]$. Therefore, the microscopic density $D(w, t)$ and $g(w, t)$ are statistically equivalent. It can be shown that if $r(a, t)$ and $v(a, t)$ are the solution of the continuity and Navier–Stokes equations with prescribed initial values $r(a, t_0)$ and $v(a, t_0)$, then $D(w, t)$ satisfies (17) and is the solution of Eq. (12) with the corresponding initial condition

$$D(w, t_0) = \int da\, \delta(x - r(a, t_0))\, \delta(v - v(a, t_0))$$

$$= \delta(v - v(x, t_0)) = \delta(v - u(x, t_0))$$

Equation (12) also admits regular solutions, such as the one-point p.d.f. $f_1(v, x, t)$, which represents statistically independent fluid particles through

$$f_n(w_1,..., w_n, t) = \prod_{k=1}^{n} f_1(w_k, t) \qquad \text{for all} \quad n \geqslant 1$$

In this special case and for $n = 1$, Eq. (3) is identical to Eq. (12) and the functional $\exp\{i \int dw_1\, \eta(w_1)\, f_1(w_1, t)\}$ satisfies Eq. (9).

In modern formulations of Bogolyubov's functional method, $\mathscr{L}[i\eta, t]$ is treated as the space characteristic functional of a random field associated with the (microscopic) dynamics of molecules.[18] This point of view was previously taken by Hosokawa,[19] who also gave a formal general solution of the corresponding molecular functional equation by adopting Rosen's solution method[14] for Hopf's equation.

4. CONCLUDING REMARKS

The formulation presented here can be extended to a space-time satistics by introducing

$$G[\zeta] = \left\langle \exp \int_{t_0}^{\infty} dt \int_{R^3} da\, \zeta(r(a, t), v(a, t), t) \right\rangle \tag{34}$$

where $\zeta(x, v, t)$ is the test function. $G[\zeta]$ generates the multiple-time p.d.f.'s of $u(x, t)$ for incompressible fluids and leads to a coupled hierarchy for these p.d.f.'s.[4] A statistically closed extension to open fluids (excited by random forces with a white noise Gaussian statistics) is also possible. In this case a statistical equilibrium can be maintained between the energy input of stationary stirring forces and the viscous dissipation; consequently, the time-independent solutions of equations corresponding to (3) and (9) are of physical significance. The definitions (6) and (34) are not restricted to incompressible fluids; however, the statistical dynamics of compressible fluids requires a joint generating functional of hydrodynamic fields. Finally, bounded fluids can be handled by extending Eq. (2); the pressure term in the interaction kernel (4) must include the appropriate Green's function for the prescribed domain, with the equation corresponding to Eq. (9) requiring an additional boundary condition on $G[\eta, t]$.

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